

Coherent states on the circle

G Chadzitaskos, P Luft and J Tolar

Department of Physics

Faculty of Nuclear Sciences and Physical Engineering

Czech Technical University in Prague

Břehová 7, CZ - 115 19 Prague, Czech Republic

E-mail: jiri.tolar@fjfi.cvut.cz, goce.chadzitaskos@fjfi.cvut.cz

Abstract.

We present a possible construction of coherent states on the unit circle as configuration space. In our approach the phase space is the product $\mathbb{Z} \times S^1$. Because of the duality of canonical coordinates and momenta, i.e. the angular variable and the integers, this formulation can also be interpreted as coherent states over an infinite periodic chain. For the construction we use the analogy with our quantization over a finite periodic chain where the phase space was $\mathbb{Z}_M \times \mathbb{Z}_M$. Properties of the coherent states constructed in this way are studied and the coherent states are shown to satisfy the resolution of unity.

1. Introduction

Coherent states belong to the most important tools in numerous applications of quantum theory. The problem of coherent states on the circle was investigated by S. de Bièvre [1], and also by J. A. González and M. A. del Olmo [2], based on canonical coherent states [3]. The Weil-Brezin-Zak transform of canonical coherent states on the real line was then the essential ingredient leading to a set of coherent states on the circle labeled by the variables of the cylinder $\mathbb{R} \times \mathbf{S}^1$.

A different approach was employed by C. J. Isham and J. R. Klauder [4]. They constructed coherent states on the circle by using the Euclidean group $E(2)$, which is the semi-direct product of groups \mathbb{R}^2 and $SO(2)$ (see also [5]). However, they observed that there does not exist an irreducible representation of $E(2)$ such that the resolution of unity holds. Therefore they considered only reducible representations and extended the method to the case of the n -dimensional sphere.

It should be noted that a very detailed study of deformation quantization on the cylinder as classical phase space was recently given in [6]. Their arguments confirm that in quantum theory the ‘quantum’ phase space $\mathbb{Z} \times S^1$ is involved, which is our starting point in this investigation.

2. Position and momentum operators on the circle

Let the configuration space be the unit circle. The position variable can be identified with the angle, i.e. it takes real values modulo 2π . Let x take a value from the interval $(-\pi, \pi)$. In the Dirac formalism the *position operator* is defined

$$\hat{Q} = \int_{-\pi}^{\pi} x|x\rangle\langle x|dx, \text{ with } \langle x|y\rangle = \delta(x-y).$$

The position operator \hat{Q} has continuous spectrum $x \in \langle -\pi, \pi \rangle$ with the corresponding eigenvectors $\{|x\rangle\}$. An arbitrary quantum state $|\psi\rangle$ can be expressed in the form

$$|\psi\rangle = \int_{-\pi}^{\pi} \psi(x)|x\rangle, \quad \text{where} \quad \psi(x) = \langle x|\psi\rangle.$$

It is useful to expand $\psi(x)$ in the Fourier series

$$\psi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

The function $\psi(x)$ is the wavefunction and the position operator acts on it $\hat{Q}\psi(x) = x\psi(x)$. The momentum eigenvectors are then defined via Fourier transform

$$|p\rangle = \int_{-\pi}^{\pi} e^{ipx}|x\rangle dx \tag{1}$$

and its inverse

$$|x\rangle = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx}|k\rangle. \tag{2}$$

In the Hilbert space formulation, where

$$\mathcal{H} = L^2(\mathbf{S}^1, d\varphi),$$

we shall follow our approach in [7]. The momentum operator will be defined using the unitary representation $\hat{V}(\alpha)$ of the group of rotations $U(1)$ of the unit circle,

$$[\mathbf{V}(\alpha)\psi](\beta) = \psi(\beta - \alpha), \quad \psi \in L^2(\mathbf{S}^1, d\varphi), \quad \alpha, \beta \in U(1). \tag{3}$$

Unitary operators $\mathbf{V}(\alpha)$ shift the argument of functions in $L^2(\mathbf{S}^1, d\varphi)$. The momentum operator is then by Stone's theorem

$$\hat{P} = -i \frac{d}{d\varphi}. \tag{4}$$

The position operator

$$(\hat{Q}\psi)(\varphi) = \varphi\psi(\varphi), \tag{5}$$

formally satisfies the well known commutation relation

$$[\hat{Q}, \hat{P}] = i\mathbb{I}, \tag{6}$$

but it is not well-defined on \mathcal{H} .

3. Construction of coherent states

In order to define coherent states directly by Perelomov's method [3], we should first construct a system of unitary operators labeled by elements of the group $\mathbb{Z} \times U(1)$; second, it is necessary to determine the 'vacuum' vector $|0, 0\rangle$. The system of unitary operators will be defined

$$\widehat{W}(m, \alpha) := e^{im\hat{Q}}e^{-i\alpha\hat{P}} = e^{im\hat{Q}}\mathbf{V}(\alpha), \quad \alpha \in [\pi, \pi), \quad m \in \mathbb{Z}. \tag{7}$$

The factors do not commute

$$e^{im\widehat{\mathbf{Q}}}e^{-i\alpha\widehat{\mathbf{P}}} = e^{im\alpha}e^{-i\alpha\widehat{\mathbf{P}}}e^{im\widehat{\mathbf{Q}}}, \quad \alpha \in [\pi, \pi), \quad m \in \mathbb{Z}, \quad (8)$$

but the operator $e^{im\widehat{\mathbf{Q}}}$ is now well defined on \mathcal{H} ,

$$e^{im\widehat{\mathbf{Q}}}\psi(\varphi) = e^{im\varphi}\psi(\varphi). \quad (9)$$

Here, the system of operators $\widehat{W}(m, \alpha)$ does not create a representation of group $\mathbb{Z} \times U(1)$ but thanks to the Weyl form of commutation relations we obtain a projective representation of $\mathbb{Z} \times U(1)$.

The vacuum vector $|0, 0\rangle$ will be determined in analogy with canonical coherent states on $L^2(\mathbb{R})$. The requirement that the vacuum state be an eigenvector of annihilation operator with eigenvalue 0 is written here in exponential form

$$e^{\widehat{Q}+i\widehat{P}}|0, 0\rangle = |0, 0\rangle. \quad (10)$$

Using the Baker-Campbell-Hausdorff formula the operator $e^{\widehat{Q}+i\widehat{P}}$ could be separated in the product of two operators $e^{\widehat{Q}}$ and $e^{i\widehat{P}}$ in arbitrary order. Such change will not influence the final vacuum state $|0, 0\rangle$.

Condition (10) leads to the Gauss exponential function

$$|0, 0\rangle = \mathcal{A}e^{-\frac{\varphi^2}{2}}, \quad \varphi \in [-\pi, \pi). \quad (11)$$

Hence the vacuum state is an element of our Hilbert space, $|0, 0\rangle \in L^2(\mathbf{S}^1, d\varphi)$. At $\varphi = \pm\pi$ it is continuous but its derivative has a small discontinuity ($\approx e^{-5}$). The normalization constant \mathcal{A} is given by

$$\mathcal{A} = \frac{1}{\sqrt{\int_{-\pi}^{\pi} \exp(-\varphi^2) d\varphi}} \doteq 0.751128. \quad (12)$$

The system of coherent states on $L^2(\mathbf{S}^1, d\varphi)$ is now obtained by the action of the system of operators $\widehat{W}(m, \alpha)$ on the vacuum state $|0, 0\rangle$:

$$|m, \alpha\rangle := \widehat{W}(m, \alpha)|0, 0\rangle. \quad (13)$$

The functional form of our coherent states is given by

$$|m, \alpha\rangle = \mathcal{A}e^{im\varphi}e^{-\frac{(\varphi-\alpha)^2}{2}}, \quad \varphi \in [-\pi, \pi), \quad (14)$$

i.e. for $\alpha \neq 0$ they are displaced and phased versions of (11) with a discontinuity at $\varphi = \pm\pi$.

4. Properties of our coherent states on $L^2(\mathbf{S}^1, d\varphi)$

In this section we shall check several properties of our coherent states which are known to hold for canonical coherent states on $L^2(\mathbb{R})$ [3, 8]. First of all, we shall look at the resolution of unity, i.e. we shall prove the following equality for our coherent states:

$$\sum_{k \in \mathbb{Z}} \int_{\mathbf{S}^1} |k, \alpha\rangle \langle k, \alpha| d\alpha = c\widehat{I}, \quad (15)$$

where c is a constant. Thus for an arbitrary normalized vector η from our Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ the identity

$$\sum_{k \in \mathbb{Z}} \int_{\mathbf{S}^1} \langle \eta | k, \alpha \rangle \langle k, \alpha | \eta \rangle d\alpha = c \hat{I}, \quad (16)$$

should hold. Since the inner product of $|\eta\rangle$ with some coherent state $|k, \alpha\rangle$ has the integral form

$$\langle k, \alpha | \eta \rangle = \mathcal{A} \int_{\mathbf{S}^1} e^{-ik\varphi} e^{-\frac{(\varphi-\alpha)^2}{2}} \eta(\varphi) d\varphi, \quad (17)$$

one can calculate the constant c with the result

$$c = 2\pi, \quad (18)$$

so that the resolution of unity for our set of coherent states takes the form

$$\sum_{k \in \mathbb{Z}} \int_{\mathbf{S}^1} |k, \alpha\rangle \langle k, \alpha| d\alpha = 2\pi \hat{I}. \quad (19)$$

The next properties to be checked concern the overlaps of our (normalized) coherent states given by inner products in $L^2(\mathbf{S}^1, d\varphi)$. Here it is necessary to correctly realize the way how the operator $\mathbf{V}(\alpha) = \exp(-i\alpha\hat{P})$ acts on the Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ when the circle \mathbf{S}^1 — our configuration space — is identified with the interval $[-\pi, \pi)$. Then the action of operator $e^{-i\alpha\hat{P}}$ on function $\psi(\varphi) \in L^2(\mathbf{S}^1, d\varphi)$ for $\alpha \in [0, \pi]$ has the form :

$$e^{-i\alpha\hat{P}} \psi(\varphi) = \begin{cases} \psi(\varphi - \alpha) & \varphi \in [-\pi + \alpha, \pi], \\ \psi(\varphi - \alpha + 2\pi) & \varphi \in [-\pi, -\pi + \alpha]. \end{cases} \quad (20)$$

For $\alpha \in [-\pi, 0]$ we have

$$e^{-i\alpha\hat{P}} \psi(\varphi) = \begin{cases} \psi(\varphi - \alpha) & \varphi \in [-\pi, \pi + \alpha], \\ \psi(\varphi - \alpha - 2\pi) & \varphi \in [\pi + \alpha, \pi]. \end{cases} \quad (21)$$

Note that one has to consider addition modulo 2π in the argument of function ψ . This is the reason why the inner product *cannot* be calculated simply according to

$$\langle m, \alpha | n, \beta \rangle = \mathcal{A}^2 \int_{-\pi}^{\pi} e^{-i\varphi(n-m)} e^{-\frac{(\varphi-\alpha)^2}{2}} e^{-\frac{(\varphi-\beta)^2}{2}} d\varphi. \quad (22)$$

From now on we shall restrict ourselves only to the cases when α and β are non-negative:

$$\alpha \in [0, \pi], \quad \beta \in [0, \pi]. \quad (23)$$

Without loss of generality we may also assume

$$\beta \geq \alpha. \quad (24)$$

Taking into account (20) and (21), we have to split the inner product of two coherent states into two terms

$$\langle m, \alpha | n, \beta \rangle = \mathcal{A}I_1(\alpha, \beta, n - m) + \mathcal{A}I_2(\alpha, \beta, n - m), \quad (25)$$

where

$$I_1(\alpha, \beta, n - m) := \int_{\alpha - \pi}^{\beta - \pi} e^{i\varphi(n-m)} e^{-\frac{(\varphi-\alpha)^2}{2}} e^{-\frac{(\varphi-\beta+2\pi)^2}{2}} d\varphi \quad (26)$$

and

$$I_2(\alpha, \beta, n - m) := \int_{\beta - \pi}^{\pi + \alpha} e^{i\varphi(n-m)} e^{-\frac{(\varphi-\alpha)^2}{2}} e^{-\frac{(\varphi-\beta)^2}{2}} d\varphi. \quad (27)$$

In order to evaluate the integrals $I_1(\alpha, \beta, n - m)$ and $I_2(\alpha, \beta, n - m)$, we use the error function of a complex variable z ,

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_{\Gamma(z)} e^{-\psi^2} d\psi; \quad (28)$$

here $\Gamma(z)$ denotes an arbitrary continuous path of finite length which connects the origin $0 \in \mathbb{C}$ with a complex number $z \in \mathbb{C}$.

The integral $I_1(\alpha, \beta, n - m)$, after the substitution

$$\omega = \varphi + \pi - \frac{\alpha + \beta}{2}, \quad (29)$$

takes the form

$$\begin{aligned} I_1(\alpha, \beta, n - m) &= e^{-(\frac{\beta-\alpha}{2})^2 - \pi} e^{i(\frac{\alpha+\beta}{2} - \pi)(n-m)} \times \\ &\times \int_{\frac{\alpha-\beta}{2}}^{\frac{\beta-\alpha}{2}} e^{i\omega(n-m)} e^{-\omega^2} d\omega \end{aligned} \quad (30)$$

which leads to the formula

$$\begin{aligned} I_1(\alpha, \beta, n - m) &= \left(-\frac{\sqrt{\pi}}{2}\right) e^{-(\frac{\beta-\alpha}{2})^2 - \pi} e^{i(\frac{\alpha+\beta}{2} - \pi)(n-m)} e^{-\frac{(n-m)^2}{4}} \times \\ &\times \left[\operatorname{erf}\left(\frac{\alpha - \beta}{2} + \frac{i(n-m)}{2}\right) \right. \\ &\left. + \operatorname{erf}\left(\frac{\alpha - \beta}{2} - \frac{i(n-m)}{2}\right) \right]. \end{aligned} \quad (31)$$

The other integral $I_2(\alpha, \beta, n - m)$, after the substitution $\omega = \varphi - \frac{\alpha+\beta}{2}$, yields

$$\begin{aligned} I_2(\alpha, \beta, n - m) &= \left(-\frac{\sqrt{\pi}}{2}\right) e^{-(\frac{\beta-\alpha}{2})^2} e^{i(\frac{\alpha+\beta}{2})(n-m)} e^{-\frac{(n-m)^2}{4}} \times \\ &\times \left[\operatorname{erf}\left(\frac{\alpha - \beta}{2} - \pi + \frac{i(n-m)}{2}\right) \right. \\ &\left. + \operatorname{erf}\left(\frac{\alpha - \beta}{2} - \pi - \frac{i(n-m)}{2}\right) \right]. \end{aligned} \quad (32)$$

We have to admit that, unfortunately, we do not see any way how to further simplify the above analytic expressions of the integrals $I_1(\alpha, \beta, n - m)$ and $I_2(\alpha, \beta, n - m)$ to see whether the coherent states are mutually non-orthogonal. However, we have numerically computed absolute values of the inner products for many pairs of coherent states and have drawn the graphs for several different values of $n - m$ fixed in each graph. It was apparent — for all plotted cases — that the overlap never vanishes. Because of restriction on the length of the contribution we intend to publish a detailed account of the matter including a number of plots confirming this property.

The next checked quantities are the expectation values of the position operator in the coherent states $|m, \alpha\rangle$ for $\alpha \geq 0$. Explicitly we have

$$\langle m, \alpha | \hat{Q} | m, \alpha \rangle = \mathcal{A}^2 \int_{-\pi}^{-\pi+\alpha} \varphi e^{-(\varphi-\alpha+2\pi)^2} d\varphi + \mathcal{A}^2 \int_{-\pi+\alpha}^{\pi} \varphi e^{-(\varphi-\alpha)^2} d\varphi, \quad (33)$$

or, using the error function,

$$\langle m, \alpha | \hat{Q} | m, \alpha \rangle = \alpha - \mathcal{A}^2 \sqrt{\pi^3} (\text{erf}(\pi) - \text{erf}(\pi - \alpha)). \quad (34)$$

One observes that for α positive the expectation values do not depend on m (the same result is obtained also for negative values of α). It is interesting that the expectation value of position is nonlinear in α for $\alpha \neq 0$. It is clear that for $\alpha = 0$ the quasi-Gaussian (11) is symmetric around $\varphi = 0$, so the expectation value is an integral of an odd function, evidently equal to zero. However, if $\alpha \in (0, \pi]$, the displaced quasi-Gaussian is no more symmetric around $\varphi = 0$ and the integration leads to a deviation depending on a difference of error functions (34); maximal deviation is attained for $\alpha = \pi$.

Important among the checked quantities are the expectation values of the momentum operator in coherent states $|m, \alpha\rangle$. The explicit form

$$\begin{aligned} \langle m, \alpha | \hat{P} | m, \alpha \rangle &= \mathcal{A}^2 \int_{-\pi}^{-\pi+\alpha} (m + i(\varphi - \alpha + 2\pi)) e^{-(\varphi-\alpha+2\pi)^2} d\varphi \\ &+ \mathcal{A}^2 \int_{-\pi+\alpha}^{\pi} (m + i(\varphi - \alpha)) e^{-(\varphi-\alpha)^2} d\varphi \end{aligned}$$

can be simplified into the formula

$$\langle m, \alpha | \hat{P} | m, \alpha \rangle = m, \quad (35)$$

which could be anticipated by analogy with the canonical coherent states on $L^2(\mathbb{R})$.

Finally, the expectation values of the square of momentum operator in coherent states $|m, \alpha\rangle$ are determined by computing the integrals

$$\begin{aligned} \langle m, \alpha | \hat{P}^2 | m, \alpha \rangle &= \mathcal{A}^2 \int_{-\pi}^{-\pi+\alpha} [1 + (m + i(\varphi - \alpha + 2\pi))^2] e^{-(\varphi-\alpha+2\pi)^2} d\varphi \\ &+ \mathcal{A}^2 \int_{-\pi+\alpha}^{\pi} [1 + (m + i(\varphi - \alpha))^2] e^{-(\varphi-\alpha)^2} d\varphi \end{aligned} \quad (36)$$

with the result

$$\langle m, \alpha | \hat{P}^2 | m, \alpha \rangle = m^2 + \frac{1}{2} + \mathcal{A}^2 \pi e^{-\pi^2} \quad (37)$$

which does not depend explicitly on α .

5. Conclusion

For our system of coherent states their overlaps and matrix elements were expressed analytically, then some calculated numerically or evaluated with the help of MATHEMATICA. The property of resolution of unity was shown to hold.

Acknowledgements

The support by the Ministry of Education of Czech Republic (projects MSM6840770039 and LC06002) is acknowledged. The authors are grateful to the referee for several constructive remarks which helped to improve the presentation.

References

- [1] de Bièvre S 1989 Coherent states over symplectic homogenous spaces, *J Math Phys* **30** 1401–1407
- [2] González J A and del Olmo M A 1998 Coherent states on the circle and quantization *J Phys A: Math Gen* **31** 884
- [3] Perelomov M A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
- [4] Isham C J and Klauder J R 1991 Coherent states for n -dimensional Euclidean groups $E(n)$ and their application *J Math Phys* **32**(3) 607–620
- [5] Nieto L M, Atakishiyev N M, Chumakov S M and Wolf K B 1998 Wigner distribution function for Euclidean systems *J Phys A: Math Gen* **31** 3875–3895
- [6] González J A, del Olmo M A and Tosiek J 2003 Quantum mechanics on the cylinder *J Opt B: Quantum Semiclass Opt* **5** S306–S315
- [7] Tolar J and Chadzitaskos G 1997 Quantization on \mathbf{Z}_M and coherent states over $\mathbf{Z}_M \times \mathbf{Z}_M$ *J Phys A: Math Gen* **30** 2509–2517
- [8] Klauder J R and Skagerstam B S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)